Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

A Unified Approach to Equilibrium Existence in Discontinuous Strategic Games

Philippe BICH, Rida LARAKI

2012.40

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/bandeau-haut/documents-de-travail/
ISSN : 1955-611X

A Unified Approach to Equilibrium Existence in Discontinuous Strategic Games*

(Preliminary Version)

Philippe Bich[†] and Rida Laraki[‡]

June 8, 2012

Abstract

Several relaxations of Nash equilibrium are shown to exist in strategic games with discontinuous payoff functions. Those relaxations are used to extend and unify several recent results and link Reny's better-reply security condition [Reny, P.J. (1999). On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games. *Econometrica*, **67(5)**, 1029-1056.] to Simon-Zame's endogenous tie-breaking rules [Simon, L.K. and Zame, W.R. (1990). Discontinuous Games and Endogenous Sharing Rules. *Econometrica*, **58**, 861-872.].

JEL classification: C02, C62, C72.

Keywords: Discontinuous games, Nash equilibrium, Reny equilibrium, better-reply security, endogenous sharing rule, quasi equilibrium, finite deviation equilibrium, symmetric games.

1 Introduction

Many classical problems in economics can be formulated as games with continuous strategy spaces and discontinuous payoff functions. Examples abound: price and spatial competitions, auctions, Nash-demand, preemption, war of attrition, and general equilibrium among many others.

To study those problems, standard equilibrium existence results, such as Nash [30, 31] and Glicksberg [14], fail. Under some "endogenous" tie-breaking rules on the discontinuity points, some of those games admit an equilibrium. Under other rules, equilibria do not exist. For instance, in a standard Bertrand game played by two firms with heterogeneous costs, a pure Nash equilibrium exists only when the ties are broken in favor of the lower-cost firm. Two natural questions arise and have been addressed in the literature. First, under which conditions does an equilibrium exist? Second, when a game does not admit an equilibrium, should another solution concept be used instead, or is there some endogenous tie-breaking rule under which an equilibrium does exist?

The first question is not only theoretical. For many games, such as pay-your-bid multi-unit auction, an equilibrium exists but it cannot be constructed explicitly (Reny [34]). Consequently, various "well behaving" conditions at discontinuities have been established in the literature, initiated by the seminal papers of Dasgupta and Maskin ([11] and [12]). A great achievement has been established by Reny [34]. He proved that any *better-reply secure* BRS quasiconcave compact game admits a pure Nash equilibrium. Reny's BRS condition requires that whenever a strategy profile x is not a Nash equilibrium, then for any limit vector payoff $v = \lim_{x_n \to x} u(x_n)$, there is a player i that can "secure" more than v_i (i.e. he has a

*We thank Panayotis Mertikopoulos and Christina Pawlowitsch for their useful comments. This research was supported by grant ANR-10-BLAN 0112 (Laraki).

†Paris School of Economics, Centre d'Economie de la Sorbonne UMR 8174, Université Paris I Panthéon Sorbonne, 106/112 Boulevard de l'Hpital 75647 Paris Cedex 13. E-mail: bich@univ-paris1.fr

‡CNRS, Economics Department, Ecole Polytechnique, France. Part time associated with Équipe Combinatoire et Optimisation, IMJ, CNRS UMR 7586, Université Pierre et Marie Curie. E-mail: rida.laraki@polytechnique.edu

deviation that guarantees strictly more than v_i for all strategy profiles y_{-i} in a neighborhood of x_{-i}). Reny's paper has induced a large and still extremely active research agenda ([5], [3], [7], [8], [27], [35] and [36]).

To the second question–which relaxation may be used when the model fails to have an equilibrium?– two independent answers have been given. One of the most used relaxation is *approximated equilibrium.* It is a limit point of ϵ-Nash equilibria as ϵ goes to 0. It exists in many standard games such as Bertrand duopoly in pure strategies (the firm with the lowest cost prices slightly below the cost of her opponent) and exists in mixed strategies in undiscounted two player stochastic games [41, 42] as well as timing games [25]. Surprisingly, approximated equilibrium is much less used in economics applications, perhaps because there is but one existence result, that of Prokopovych [33]. His conditions apply to Bertrand duopoly but are of no help in stochastic and timing games where, in general, approximated equilibria are explicitly constructed.

One of the main aims of this paper is to define simple conditions for the existence of an approximated equilibrium that explain why all the models listed above admit such a solution. We are convinced that Reny's [34] BRS approach (which essentially establishes conditions on the closure of the graph $\{(x, u(x)), x \in X\}$ of the game) offers a key direction: any sequence of ϵ_n-equilibria induces an element $(x, v) = \lim_n (x_{\epsilon_n}, u(x_{\epsilon_n}))$ in the closure of the graph. The limit vector payoff v is a useful information that partially captures how x_ϵ must tend to x. Consequently, a first step for establishing a useful "fixed point" theorem for the existence of an approximated equilibrium is to get a deeper understanding of Reny's results by means of concepts that pertain to the closure of the graph of the game. For, one needs deeper understanding of the BRS's condition and, more importantly, one should establish a topological proof of Reny's result. Actually, the original proof [34] is non-standard and proceeds by contradiction to construct a sequence of approximations by a family of Euclidean games on polytopes.

Our search led us naturally to explore a possible link with endogenous tie-breaking rules: if (x, v) is an approximated equilibrium, and x is a discontinuity point, v will be the endogenous tie-breaking rule at x. The exploration of this intuition gives rise to new questions and so new solution concepts both in pure and mixed strategies. At the end, we have a consistent and unified picture with several relaxations. Interestingly, most of them could be encompassed for games in mixed strategies (finite deviation equilibrium), and when the game is symmetric (light equilibrium).

The idea of endogenous tie-breaking rules goes back to Simon-Zame [38]. As already observed by Dagsputa and Maskin [12] and Reny [34], in most applications, discontinuities are situated in a low-dimensional subspace of strategy profiles (firms or players choosing the same price, location, bid, acting time, etc). In some of those games, the *exogenously* given tie-breaking rule leads to no pure Nash equilibrium (Bertrand duopoly and Hotelling) or no mixed equilibrium (location games [39], 3-player timing games [25], first price auction [13] and second price auction [20] among others). Often, however, equilibrium may be restored if the tie-breaking rule is chosen *endogenously* (see Maskin-Riley [26] and Kim-Che [24]).

Simon and Zame [38] argue convincingly that "*payoffs should be viewed as only partially determined, and that whenever the economic nature of the problem leads to indeterminacies, the sharing rule should be determined endogenously.*" They define a *solution* of the original game G to be a mixed Nash equilibrium of a modified game $\tilde{G}$, close to G (roughly, the payoffs of the new game $\tilde{G}$ are convex combinations of limit payoffs of G), and prove that any compact game admits a solution. The result has been extended to strategic games with incomplete information by Jackson et al. [19].

The approaches of both Reny and Simon and Zame are widely used in economics and specifically auctions to prove existence of an equilibrium or to restore its existence. This leads Jackson and Swinkel [21] to ask if "*these approaches turn out to be related*".

To establish our first formal link between the two main approaches in pure strategies, we introduce two *equivalent* relaxations of Nash equilibrium: both defined on the closure of the graph of the game. We show that these relaxations of Nash equilibrium exist in any quasiconcave compact game. A *Reny equilibrium* is a strategy profile x and a limit vector payoff v such that no player i can secure more than v_i. A *sharing rule equilibrium* is a strategy profile x and a limit vector payoff v where x is a pure Nash equilibrium of an auxiliary game $\tilde{G}$ with associated payoff v where utilities are modified only at discontinuity points with respect to two conditions (i) the new game is close to the original (i.e. its graph is included in the *closure* of the graph of G) and (ii) utility functions in $\tilde{G}$ are bounded below by utility levels that can be secured in G. Condition (i) is a strengthening of the notion of closeness considered in Simon and Zame[38] (no convex hull is needed). Moreover, they do not require any lower bound on utilities as our condition (ii) does. This answers an open question asked by Jackson et al [19]: "*We should emphasize that our results concern only the existence of solutions [sharing rules] in mixed strategies; we have little to say about the existence of solutions [sharing rules] in pure strategies*".

Our findings unify Reny's and Simon and Zame's results in pure strategies. We show that sharing rule and Reny equilibria coincide. Furthermore, we show that Nash and approximated equilibria are always Reny equilibria and, a game is BRS if and only if Reny and Nash equilibria coincide.

Finally, the existence of Reny equilibrium in any quasiconcave compact game leads automatically to a new class of games for which an approximated equilibrium does exist. Those are quasiconcave compact games satisfying a weaker form of BRS: whenever (x, v) is not an approximated equilibrium, there is a player i that can "secure" more than v_i. This is shown to easily recover Prokopovych's condition [33].

To establish a formal link between Reny and Simon and Zame in mixed strategies, we introduce a refinement of Reny equilibrium. A *finite deviation equilibrium* (FDE) is a mixed strategy profile m and an associated limit payoff vector v such that for any finite discretization D of the game and any neighborhood W of (m, v), there is $(m^D, u(m^D))$ in W such that no player i has a deviation d_i in D that increases his payoff against the mixed profile m^D_{-i}. The concept is closely related to the finite deviation property recently introduced by Reny [35, 36] who proves it to improve the better-reply security condition. A FDE is proven to exist in any compact-metric game, to strictly refine Simon and Zame's solution, and to not exist for games in pure strategies.

To restrict as far as possible equilibrium conditions to capture the recent literature of discontinuous games in pure strategies, we introduce another refinement of Reny equilibrium, mainly related to a condition in Barelli and Soza [3]. A *quasi-equilibrium* is shown to exist in any quasiconcave compact game and to have its equivalent sharing rule version (where only the lower bound on payoffs in condition (ii) is improved). The existence proof of a quasi-equilibrium is a consequence of a new selection lemma combined with a standard fixed point theorem à la Brouwer. This allows us to encompass most of the recent extensions of Reny's BRS condition for a discontinuous game to admit a Nash equilibrium ([5], [3], [7] and [8] among others).

Finally, other related solution concepts (single deviation, weak, tight and pseudo equilibrium) are explored in pure strategies. Some are shown to always exist (tight and weak equilibrium), some do not (single deviation and pseudo). This is used to show that our refinements are sharp in pure and mixed strategies.

For symmetric, quasiconcave and compact games, we introduce the concept of *light equilibrium*. We show it to exist and to refine symmetric single deviation, weak, tight, Reny and quasi equilibrium concepts. This provides extremely sharp conditions for existence of symmetric Nash or approximated equilibria. The proof is short and completely topological. This proves and improves Reny's [34] BRS condition for symmetric games.

Our three main objectives are now achieved: (1) a short topological proof for Reny's result and its recent extensions, (2) a unified picture of the literature in discontinuous games and (3) the identification of a large class of games for which an approximated equilibrium exists. This opens the door to a new and challenging research agenda for potential applications in stochastic games, auctions and industrial organization.

The paper is organized as follows. Section 2 describes the environment, notations and the main approaches of the literature (Reny [34], Simon Zame [38] and Prokopovych [33]). Section 3 is devoted to unify Reny's, Simon Zame's and Prokopovych's results in pure strategies with the introduction of a new equilibrium concept (that we call Reny equilibrium). Section 4 is devoted to link them for games in mixed strategies throughout the introduction of the concept of finite deviation equilibrium. Section 5 presents other new concepts that refine the previous one, improves our concepts in case of symmetric games, and shows the sharpness of the results. Section 6 concludes. Appendix contains the most technical proofs.

2 Three approaches to discontinuous games

This section recalls three approaches to Nash existence in discontinuous games, represented by better-reply secure games [34], approximated equilibrium [33] and endogenous tie-breaking rules [38].

The paper considers games in strategic form with a finite set N of players.[1] The pure strategy set X_i of each player $i \in N$ is a non-empty, compact subset of a (not necessarily Hausdorff or locally convex) topological vector space. Each player i has a bounded payoff function $u_i : X = \prod_{i\in N} X_i \to \mathbb{R}$. A strategic form game G is thus a couple $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$. Whenever G satisfies the above assumptions, it is called *compact*. However, some results will need the space to be metric, Hausdorff, or locally convex.

As usual, for every $x \in X$ and $i \in N$, denote $x_{-i} = (x_j)_{j\neq i}$ and $X_{-i} = \Pi_{j\neq i} X_j$.

G is *quasiconcave* if for every player $i \in N$, X_i is convex and every strategy $x_{-i} \in X_{-i}$, the mapping $u_i(\cdot, x_{-i})$, defined on X_i, is quasiconcave.

G is *continuous* if for every player i, u_i is a continuous mapping, where X is endowed with the product topology. For every game G, $\Gamma = \{(x, u(x)) : x \in X\}$ denotes its graph. Finally, let $\overline{\Gamma}$ be the closure of Γ and $\overline{\Gamma}_x = \{v \in \mathbb{R}^N : (x, v) \in \overline{\Gamma}\}$ be the x-section of $\overline{\Gamma}$. A *pure Nash equilibrium* of G is a strategy profile $x = (x_i)_{i\in N} \in X$ such that for every player $i \in N$ and every deviation $d_i \in X_i$, $u_i(d_i, x_{-i}) \leq u_i(x)$.

2.1 Better-reply secure games

In many discontinuous games, a Nash equilibrium exists. Reny's [34] existence result explains why.

For every $x \in X$ and every player i, define $\underline{u_i}(d_i, x_{-i})$ the largest payoff player i can obtain when he plays d_i and the other players may deviate slightly from x_{-i}: formally,

$$\underline{u_i}(d_i, x_{-i}) = \liminf_{x'_{-i}\to x_{-i}} u_i(d_i, x'_{-i}) = \sup_{V\in\mathcal{V}(x_{-i})} \inf_{x'_{-i}\in V} u_i(d_i, x'_{-i}),$$

where $\mathcal{V}(x_{-i})$ denotes the set of neighborhoods of x_{-i}. Hence, $\underline{u_i}(x_i, x_{-i})$ is just the lower semicontinuous regularization of $u_i(x_i, .)$ with respect to x_{-i}. (The notation $\liminf_{x'_{-i}\to x_{-i}}$ is misleading since the strategy space in not necessary metric: convergence should be understood is the sense of nets.)

Definition 1 *A game is better-reply secure (BRS) if for every $(x, v) \in \overline{\Gamma}$ such that x is not a Nash equilibrium, there exists $i \in N$ and a deviation $d_i \in X_i$ such that $\underline{u_i}(d_i, x_{-i}) > v_i$.*

Theorem 2 (Reny [34]) *Any better-reply secure quasiconcave compact game G admits a Nash equilibrium.*

Reny [34] provides simple sufficient conditions under which a game is BRS.

Definition 3 *A game is payoff secure if for every $x \in X$, every $\varepsilon > 0$ and every $i \in N$, there exists $d_i \in X_i$ and $V_{x_{-i}}$, a neighborhood of x_{-i}, such that $u_i(d_i, x'_{-i}) \geq u_i(x) - \varepsilon$ for every $x'_{-i} \in V_{x_{-i}}$.*

Equivalently, if for every i and x_{-i}:

$$\sup_{d_i\in X_i} u_i(d_i, x_{-i}) = \sup_{d_i\in X_i} \underline{u_i}(d_i, x_{-i})$$

.

Definition 4 *A game is reciprocally upper semicontinuous if whenever $(x, v) \in \overline{\Gamma}$ and $u(x) \leq v$ then $u(x) = v$.*

The last condition holds if the sum of utilities is upper semicontinuous. Reny [34] proves that any payoff-secure and reciprocally upper semicontinuous game is better-reply secure.

[1] Games with an infinite number of players may naturally be used to study existence of perfect equilibria of a finite player extensive form game with infinitely many decision nodes as soon as the single deviation property holds. In that case, finding a perfect equilibrium is equivalent to finding a perfect equilibrium of a strategic form game with infinitely many players, one for each decision node. Most of our solution concepts in pure strategies extend to infinitely many players (see section 5 and appendix).

2.2 Approximated equilibrium

A natural and widely used relaxation of Nash is the following.

Definition 5 *A pair $(x, v) \in \overline{\Gamma}$ is an approximated equilibrium if there exists a sequence*[2] *$(x^n)_{n\in\mathbb{N}}$ of X such that: (i) for every $n \in \mathbb{N}$, x^n is a $\frac{1}{n}$-equilibrium, that is: for every player $i \in N$ and every deviation $d_i \in X_i$, $u_i(d_i, x^n_{-i}) \leq u_i(x^n) + \frac{1}{n}$. (ii) the sequence $(x^n, u(x^n))$ converges to (x, v).*

Let $V_i(x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ denotes the maximal payoff player i could obtain against x_{-i}.

Theorem 6 (Prokopovych [33]) *Any payoff-secure quasiconcave compact game G such that V_i is continuous for every i admits an approximated equilibrium.*

2.3 Endogenous tie-breaking rules

A completely different answer to discontinuities has been proposed by Simon and Zame ([38]). They prove existence of tie-breaking rules at discontinuity points so that the modified game admits a mixed Nash equilibrium.

Let $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form. It is assumed to be compact-metric, that is, the strategy sets X_i are compact and metrizable (but not necessary convex) and the utility functions are measurable and bounded.

Denote by $M_i = \Delta(X_i)$ the set of Borel probability measures on X_i (i.e. the set of mixed strategies of player i). This is also a compact-metrizable set under the weak* topology. Let $M = \Pi_i M_i$. A *mixed equilibrium* of G is a pure Nash equilibrium of its mixed extension $G' = (\{M_i\}_{i\in N}, \{u_i\}_{i\in N})$ where utilities are extended multi-linearly to M.

Definition 7 *Let G be a compact-metric game. A double (σ, q) is a solution of G if $\sigma \in M$ is a mixed equilibrium of the auxiliary game in strategic form $\tilde{G} = ((X_i)_{i\in N}, (q_i)_{i\in N})$ where the auxiliary utilities $q = (q_i)_{i\in N}$ must satisfy condition (SZ): $\forall y \in X,\ q(y) \in co\overline{\Gamma}_y$, where co stands for the convex hull.*

Condition (SZ) has implications. When u_i is continuous at x then $q_i(x) = u_i(x)$. When $\sum_i u_i(x)$ is continuous then $\sum_i q_i(x) = \sum_i u_i(x)$. This justifies our subsequent terminology sharing rule when utilities are transferable. More generally, Jackson et al. [19] interpreted a tie-breaking rule as a proxy for the outcome of an unmodeled second stage game. As example, they recall the analysis of first-price auctions for a single indivisible object. Maskin and Riley [26] adjoin to the sealed-bid stage a second stage where bidders with the highest bid in the first stage play a Vickrey auction. In the private value setting, their dominant strategy is to bid their true values. Consequently, the second stage induces a tie-breaking rule where the bidder with the highest value wins the object. A proper way of implementing the tie-breaking rule is by asking players to send to the seller, in addition of their bids (in general their strategies) their private values (in general a cheap message). The messages will be used only to break ties. Jackson et al. [19] used this interpretation to extend their result to strategic games with incomplete information.

Theorem 8 (Simon and Zame [38]) *Any compact-metric game admits a solution.*

3 Reny and sharing rule equilibrium in pure strategies

Jackson and Swinkel [21] asked whether there is a link between Simon and Zame's and Reny's results. Before providing a formal link, consider two examples.

[2] Again, the definition is stated in term of sequences, while one should use convergence in the sense of nets. The choice is made for simplicity of the presentation.

Example 9 *The California-Oregon location game is considered in the paper of Simon and Zame. The length interval* $[0,4]$ *represents an interstate highway. The strategy space of player 1 (the psychologist from California) is* $X = [0,3]$ *(representing the Californian highway stretch). The strategy space of player 2 (the psychologist from Oregon) is* $Y = [3,4]$ *(the Oregon part of the highway). The payoff function of player 1 is* $u_1(x,y) = \frac{x+y}{2}$ *if* $x < y$ *and* $u_1(3,3) = 2$*. The payoff function of player 2 is* $u_2(x,y) = 4 - u_1(x,y)$*. This game is constant-sum, discontinuous and has no Nash equilibrium. Define* $q(x) = u(x)$ *for every* $x \neq (3,3)$ *and let* $q(3,3) = (3,1)$*. The pure strategy profile* $(3,3)$ *with utility profile* q *define clearly a solution à la Simon and Zame. Here, the new sharing rule at* $(3,3)$ *has a simple interpretation: it corresponds to giving each psychologist its natural market share.*

Example 10 *In a Bertrand duopoly, two firms* $i = 1,2$ *choose prices* $p_i \in [0,a]$ $(a > 0)$*. Assume a linear demand* $a - min(p_1,p_2)$ *and marginal costs* $c_1 < c_2 < \frac{a+c_1}{2}$*. The game has no pure Nash equilibrium if one assumes that the firm charging the lowest price supplies the entire market. Nevertheless, the game is quasiconcave and compact. This game, has a solution with strategy profile* (c_2,c_2) *and with payoff profile* $q(c_2,c_2) = ((a-c_2)(c_2-c_1),0)$ *and* $q(x) = u(x)$ *elsewhere. This equilibrium profile may be related to another standard tool to circumvent the non-existence of a Nash equilibrium: just assume that there is a smallest monetary unit* $\delta > 0$*. Then, the strategy profile* $(c_2-\delta, c_2)$ *is a Nash equilibrium of this discretized game for every* $\delta > 0$ *small enough. The associated payoff vector is* $((c_2-\delta-c_1)(a-c_2+\delta),0)$*. As* δ *goes to zero, one obtains the Simon and Zame solution.*

Several remarks are of interest. Both examples yield a solution (à la Simon and Zame) in *pure strategies*, yet Simon and Zame's solution is stated in *mixed strategies*. Second, the solutions happen to be approximated equilibria in both cases (in Example 9, consider the sequence of $\frac{1}{n}$-Nash equilibria $(3-\frac{1}{n},3)$, and in Example 10, the sequence $(c_2-\frac{1}{n},c_2)$). Finally, the games are not better-reply secure and so Reny's result seems of no help to explain why. Yet, as proved below, Reny's result permits to understand why those games have a solution à la Simon and Zame in pure strategies and that this solution is indeed an approximated equilibrium.

Definition 11 *A couple* $(x,v) \in \overline{\Gamma}$ *is a sharing rule equilibrium of* G *if* x *is a Nash equilibrium of an auxiliary game* $\tilde{G} = ((X_i)_{i\in N},(q_i)_{i\in N})$*,* $v = q(x)$ *is the associated vector payoff and:*
(i) For every $y \in X$*,* $q(y) \in \overline{\Gamma}_y$*.*
(ii) For every player $i \in N$ *and every* $d_i \in X_i$*,* $q_i(d_i,x_{-i}) \geq \underline{u_i}(d_i,x_{-i})$*.*

In words, a sharing rule equilibrium is a profile x and a payoff vector v such that x is a classical Nash equilibrium of an auxiliary game and v is the associated payoff. Without conditions on $\tilde{G}$, any strategy profile could be made a Nash equilibrium of some auxiliary game. To reduce the number of solutions, the new game must be as close as possible to the original game. For, some restrictions are necessary. Condition (i) is weaker that condition (SZ): no convex hull is taken. It says that $q(y)$ could be obtained as a limit point of some $u(y_n)$ as y_n goes to y. Thus, only payoffs at discontinuity points of u are modified. Condition (ii) asks that the payoff of a player in the auxiliary game remains above the payoff he can secure in the original game. Such a lower bound is not required by Simon and Zame. The following example shows its importance.

Example 12 *Consider a one-player game who maximizes over* $[0,1]$ *the following discontinuous utility function:* $u(x) = 0$ *if* $x < 1$*, and* $u(1) = 1$*. The most natural solution is clearly the profile* $x = 1$ *yielding the highest possible payoff. But the constant payoff function* $q = 0$ *satisfies condition (i). So, any mixed strategy profile* σ *is a solution à la Simon and Zame! When condition (ii) is added,* $q = u$ *is the unique remaining utility function.*

More generally, in a one player game, the upper semicontinuous regularization of u:

$$u^{u.s.c.}(x) = \lim\sup_{x' \to x} u(x'),$$

satisfies (i) and (ii). For two players or more, the utility profile $q = (u_i^{u.s.c.})_{i\in N}$ may not satisfy (i) or even (SZ). Examples are easy to find.

The following result extends and refines Simon and Zame in pure strategies. This was an open question in Jackson et al [19].

Theorem 13 *Any quasiconcave compact game G admits a sharing rule equilibrium.*

To prove existence of sharing rule equilibrium, we introduce the following new equilibrium concept, inspired by the Reny BRS condition.

Definition 14 *A Reny equilibrium is a couple $(x, v) \in \overline{\Gamma}$ such that*

$$\forall i \in N, \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i.$$

A possible interpretation follows. Players in the game are called to play some profile x_n close to the Reny equilibrium profile x, whose payoff is $1/n$ close to v. No player has a deviation that secures strictly more than his Reny equilibrium payoff.

Theorem 15 *Any quasiconcave compact game G admits a Reny equilibrium.*

Without surprise, the proof is a logical consequence of Reny [34]:

Proof of Theorem 15. Assume, by contradiction, that there is no Reny equilibrium (and so no Nash equilibrium by Proposition 16). This implies that the game is better-reply secure. Consequently, from Reny [34], there exists a Nash equilibrium, which is a Reny equilibrium, a contradiction.
■

Proof of Theorem 13.

From Theorem 15, there exists a Reny equilibrium $(x, v) \in \overline{\Gamma}$. Then, one can build the sharing rule equilibrium as follows: for every $i \in N$ and $d_i \in X_i$, first denote by $\underline{\mathcal{S}}(d_i, x_{-i})$ the space of sequences $(x_{-i}^n)_{n\in\mathbb{N}}$ of X_{-i} converging to x_{-i} such that $\lim_{n\to+\infty} u_i(d_i, x_{-i}^n) = \underline{u}_i(d_i, x_{-i})$. Then, define $q : X \to \mathbb{R}^N$ by

$$q(y) = \begin{cases} v & \text{if } y = x, \\ \text{any limit point of } (u(d_i, x_{-i}^n))_{n\in\mathbb{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x_{-i}^n)_{n\in\mathbb{N}} \in \underline{\mathcal{S}}(d_i, x_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Now, prove that x is a sharing rule equilibrium profile associated to q. Since $(x, v) \in \overline{\Gamma}$, and by definition of q, condition (i) of definition 11 is satisfied at x. Obviously, it is satisfied at every y different from x for at least two components, and also at every (d_i, x_{-i}) with $d_i \neq x_i$, from the definition of $q(d_i, x_{-i})$ in this case. To prove condition (ii) of definition 11, let $i \in N$ and $d_i \in X_i$. If $d_i = x_i$ then $q_i(d_i, x_{-i}) = q_i(x) = v_i \geq \underline{u_i}(x)$ because (x, v) is a Reny equilibrium. If $d_i \neq x_i$ then $q_i(d_i, x_{-i}) = \underline{u}_i(d_i, x_{-i})$ so that condition (ii) holds.
■

The following proposition links Reny, Nash, approximated and sharing rule equilibria.

Proposition 16 *i) Reny and sharing rule equilibria coincide.*

ii) If $x \in X$ is a Nash equilibrium, $(x, u(x))$ is a Reny equilibrium.

iii) If $(x, v) \in \overline{\Gamma}$ is an approximated equilibrium, (x, v) is a Reny equilibrium.

iv) A game is better-reply secure if and only if Nash and Reny equilibria coincide.

Proof. Part i) is a consequence of the last proof and the definition of a sharing rule equilibrium. For ii), if $x \in X$ is a Nash equilibrium, then for every player $i \in N$ and every deviation $d_i \in X_i$, one has $u_i(x) \geq u_i(d_i, x_{-i}) \geq \underline{u_i}(d_i, x_{-i})$. For iii), if $(x, v) \in \overline{\Gamma}$ is an approximated equilibrium, then let $(x^n)_{n\in\mathbb{N}}$ be a sequence of $\frac{1}{n}$-equilibria such that $(x^n, u(x^n))$ converges to (x, v). For every $n \in \mathbb{N}$, for every player $i \in N$ and every deviation $d_i \in X_i$, one has $u_i(d_i, x_{-i}^n) \leq u_i(x^n) + \frac{1}{n}$. Passing to the infimum limit when

n tends to infinity, one obtains $\underline{u_i}(d_i, x_{-i}) \leq v_i$. To finish, iv) is a straightforward consequence of the definitions of Reny equilibrium and better-reply security.

■

This opens the door to a large class of games for which an approximated equilibrium exists. This class may be improved for every refinement of Reny equilibrium.

Definition 17 *G is approximately better-reply secure if for every $(x, v) \in \overline{\Gamma}$, if (x, v) is not an approximated equilibrium then there exists $i \in N$ and a deviation $d_i \in X_i$ such that $\underline{u_i}(d_i, x_{-i}) > v_i$.*

A different characterization is given in the appendix. From the existence of a Reny equilibrium one immediately obtains.

Theorem 18 *Any approximately better-reply secure quasiconcave compact game G admits an approximated equilibrium.*

Interestingly, Prokopovych's [33] result may easily be derived from the above theorem.

Corollary 19 *A payoff-secure compact game G such that V_i is continuous for every i is approximately better-reply secure.*

Recall that $V_i(x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i})$ and that a game is payoff secure if and only if: $V_i(x_{-i}) = \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i})$.

Proof. Take $(x, v) \in \overline{\Gamma}$ a Reny equilibrium, and prove it to be an approximated equilibrium. Since G is payoff secure at $x \in X$, one has $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) = \sup_{d_i \in X_i} u_i(d_i, x_{-i}) = V_i(x_{-i})$. From Reny equilibrium condition one has $V_i(x_{-i}) \leq v_i$. From the continuity of V_i, for every $\epsilon > 0$ and every neighborhood V_x of x, there exists $x' \in V_x$ such that $\mid v_i - u_i(x') \mid \leq \frac{\epsilon}{2}$ and $\sup_{d_i \in X_i} u_i(d_i, x'_{-i}) \leq \sup_{d_i \in X_i} u_i(d_i, x_{-i}) + \frac{\epsilon}{2}$, which completes the proof, since x' is an $\epsilon-$equilibrium.

■

The existence of a Reny equilibrium also simplifies the proof that a payoff secure and reciprocally upper semicontinuous game is better-reply secure [34].

Actually, let (x, v) be a Reny equilibrium, then $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i$. Since the game is payoff secure one obtains $\sup_{d_i \in X_i} u_i(d_i, x_{-i}) \leq v_i$. Taking $d = x$ one deduces that $u(x) \leq v$. Since the game is reciprocally upper semicontinuous one obtains that $v = u(x)$. Consequently, x is a Nash equilibrium.

Games in Examples 9 and 10 are approximately better-reply secure. The first does not satisfy Prokopovych's conditions. We have now a better understanding and a unified picture about the three important approaches for discontinuous games in pure strategies. The key is Reny equilibrium whose existence is a trivial consequence of Reny's main result [34].

4 Finite deviation equilibrium in mixed strategies

Let G be a compact metric game, G' be its mixed extension and let $\overline{\Gamma'}$ be the closure of the graph of G'. The following definition is related to a condition in Reny [35, 36].

Definition 20 *$(m, v) \in \overline{\Gamma'}$ is a finite deviations equilibrium (FDE) of G if for every open set $V^{m,v}$ that contains (m, v) and every finite sets $D_i \subset X_i$, $i = 1, ..., N$, there exists $m' \in M$ such that:*

- $(m', u(m')) \in V^{m,v}$;
- $\forall i, \forall d_i \in D_i : u_i(d_i, m'_{-i}) \leq u_i(m')$.

This relaxed equilibrium concept is extremely intuitive. It says that m with payoff u is "almost" a usual Nash mixed equilibrium since for any finite set of pure deviations $D_i \subset X_i$, there is m' as close as one wishes to m with a payoff $u(m')$ as close as one wishes to v such that no player i has a deviation against m' when he is restricted to use only deviations in D.

Theorem 21 *Any compact metric game G admits a FDE.*

Proof. Let us construct explicitly a particular one using finite approximations. Let $\mathcal{D}$ the set of all finite subsets $\Pi_{i\in N}D_i$ of X. Consider the inclusion relation on $\mathcal{D}$ (which is a reflexive, transitive and binary): then, each pair $\Pi_{i\in N}D_i$ and $\Pi_{i\in N}D'_i$ in $\mathcal{D}$ has an upper bound $\Pi_{i\in N}(D_i\cup D'_i)$ in $\mathcal{D}$. The couple $(\mathcal{D},\subset)$ is called a directed set. To every $D=\Pi_{i\in N}D_i\in\mathcal{D}$, one can associate $(m^D,u(m^D))$ where m^D is a mixed equilibrium of the mixed extension of D. This defines a mapping from $\mathcal{D}$ to $\overline{\Gamma'}$, called a net (of $\overline{\Gamma'}$). Then, a limit point $(m,v)\in\overline{\Gamma'}$ of the net $(m^D,u(m^D))_{D\in\mathcal{D}}$ by the following property: for every neighborhood $V_{m,v}$ of (m,v) and every $D=\Pi_{i\in N}D_i\in\mathcal{D}$, there exists $D'\in\mathcal{D}$ with $D\subset D'$ such that $(m^{D'},u(m^{D'}))\in V_{m,v}$. Such a limit point (m,v) exists because every net in a compact set has a limit point, and is clearly a FDE. ■

Observe that any Nash equilibrium is a FDE: this is a relaxation of the concept of Nash. Not surprising, a finite deviation equilibrium is a refinement of Reny equilibrium.

Proposition 22 *Any finite deviation equilibrium (m,v) is a Reny equilibrium.*

Proof. A finite deviation equilibrium implies that for every $d\in X$ and every V a neighborhood of (m,v), there exists $(m',u(m')\in V$ such that for any i, $u_i(d_i,m'_{-i})\leq u_i(m')$. Tending V to $\{(m,v)\}$ yields a Reny equilibrium. ■

Proposition 23 *Any finite deviation equilibrium (m,v) is a solution à la Simon Zame.*

The converse is false as shown in example 12.

Proof. Since we are in a metric game, there exists a sequence of finite sets $\Pi_{i\in N}D_i^n$ that converges to X and a sequence of profiles m^n such that $(m^n,u(m^n))$ converges (m,v), and such that $\forall i,\forall d_i\in D_i^n$: $u_i(d_i,m^n_{-i})\leq u_i(m^n)$. Then we follow Simon and Zame's [38] proof. ■

5 Other concepts and link with literature

To cover some recent extensions of Reny's [34] Nash existence result in pure strategies described below (Barelli and Soza [3] and Reny [35]), one needs to refine Reny (and so sharing rule) equilibrium.

In definition 11 of sharing rule equilibrium the mapping q_i is required to lay above the security utility level $\underline{u_i}$. The higher the lower bound, the sharper is the concept. This is the key idea of the next improvements. Other directions are explored, in particular symmetric games.

5.1 Strong sharing rules

Define the following regularization of u_i, following a clever idea of Barelli and Soza [3]. It allows the deviating player to choose regular deviation maps that react to small perturbations of the opponents.

$$\underline{\underline{u_i}}(d_i,x_{-i}):=\sup_{U\in\mathcal{V}(x_{-i})}\sup_{\Phi_i\in W_U(d_i,x_{-i})}\inf_{x'_{-i}\in U,d'_i\in\Phi_i(x'_{-i})}u_i(d'_i,x'_{-i}),\tag{1}$$

where $W_U(d_i,x_{-i})$ is the set Kakutani multi-valued mappings from U to X_i (i.e. having a closed graph and non-empty and convex values) such that $d_i\in\Phi_i(x_{-i}))$. The difference with $\underline{u_i}$ is that player i can now adapt his deviation d_i regularly to small perturbations of the opponents. Of course, $\underline{\underline{u_i}}(d_i,x_{-i})\geq\underline{u_i}(d_i,x_{-i})$ since one can take constant mapping $\Phi_i(y)=d_i$ in the supremum in Equation 1. Theorem 13 concerning sharing rules could be improved as follows.

A game has the Kakutani [22] property if any correspondence from X to X that has a closed graph and non-empty convex values admit a fixed point. This is the case for example if the topological vector space is locally convex and Hausdorff [1]. Recent results in topology proves it for complete metric spaces [23].

Theorem 24 *Any quasiconcave compact game G that have the Kakutani property admits a **strong sharing rule** equilibrium: that is, there exists $(x, v) \in \overline{\Gamma}$ such that x is a pure Nash equilibrium of some game $\tilde{G} = ((X_i)_{i \in N}, (q_i)_{i \in N})$, with associated payoff $v = q(x)$ and the following two conditions hold:*

(i) for every $y \in X$, $q(y) \in \overline{\Gamma}_y$.

(ii) For every player $i \in N$ and every $d_i \in X_i$, $q_i(d_i, x_{-i}) \geq \underline{\underline{u_i}}(d_i, x_{-i})$.

To prove this theorem, we need to introduce the corresponding improvement of Reny's equilibrium, presented in the next section.

5.2 Quasi equilibrium

Theorem 25 *Any quasiconcave compact game G that has the Kakutani property admits a **quasi equilibrium**: that is, there is $(x, v) \in \overline{\Gamma}$: $\forall i \in N$, $\sup_{d_i \in X_i} \underline{\underline{u_i}}(d_i, x_{-i}) \leq v_i$, or equivalently if for every i:*

$$\sup_{V \in \mathcal{V}(x_{-i})} \sup_{\Phi_i \in W_V} \inf_{x'_{-i} \in V, d_i \in \Phi_i(x'_{-i})} u_i(d_i, x'_{-i}) \leq v_i,$$

where W_V is the set of Kakutani multi-valued mapping from V to X_i.

A game G is *generalized better-reply secure* if: whenever $(x, v) \in \overline{\Gamma}$ and x is not a Nash equilibrium, there is i and d_i such that: $\underline{\underline{u_i}}(d_i, x_{-i}) > v_i$.

This means that player i admits locally (around x_{-i}) a regular deviation mapping that guarantees v_i (i.e. there is a neighborhood V of x_{-i} and a Kakutani mapping Φ_i from V to X_i such that forall $x'_{-i} \in V$ and all $d_i \in \Phi_i(x'_{-i})$, one has $u_i(d_i, x'_{-i}) > v_i$). In the Reny BRS condition, d_i is restricted to be the same in the neighborhood of x_{-i}, here it can vary but in a regular way.

Corollary 26 *[Barelli and Soza [3]] Any generalized better-reply secure quasiconcave compact game G that has the Kakutani property admits a Nash equilibrium*

This result extends McLennan et al. [27] (in the quasiconcave case) and Carmona [7]. We improve it in the next section throughout the new concept of tight equilibrium, whose completely topological proof is established in the appendix.

The following quasiconcave compact 2-player game is an illustration of the difference between quasi and Reny equilibria, or equivalently between sharing and strong sharing rules. See Barelli and Soza [3] for other examples and applications.

Example 27 *The strategy spaces are $X = Y = [0, 1]$. Player 1's payoff is $u_1(x, y) = -1$ if $x \neq y$ and $u_1(x, y) = 1$ if $x = y$. Player 2's payoff is $u_1(x, y) = -1$ if $y < 1$, $u_1(x, 1) = 1$ if $x < 1$ and $u_1(1, 1) = -2$. There is no Nash equilibrium, and only one sharing rule equilibrium profile $(1, 1)$. However, there is two possible associated utility functions q's: for $(x, y) = (1, 1)$ we have two possibilities: $q(1, 1) = (-1, 1)$ or $q'(1, 1) = (1, -1)$. For $(x, y) \neq (1, 1)$, there are no choice and one should define $q'(x, y) = q(x, y) = u(x, y)$. In both cases condition (i) is satisfied and the profile $(1, 1)$ is the only Nash equilibrium of the auxiliary game. But the choice q' seems better than q, since in this case no player sees his payoff decreasing compared to the original game. In fact, only q' satisfies condition (ii) the strong sharing rule equilibrium because $\underline{\underline{u_1}}(1, 1) = 1$. Since $\underline{u_1}(1, 1) = -1$, q satisfies condition (ii) of the weak version of sharing rule equilibrium. Consequently, the strong version is a strict refinement of weak version.*

5.3 Single deviation equilibrium

A natural question arises. Since finite deviation equilibrium refines Reny equilibrium in mixed strategies, does such an equilibrium exists in pure strategies? The following concept also refines Reny equilibrium and is a relaxation of finite deviation equilibrium.

Definition 28 *$x \in X$ is a single deviations equilibrium (SDE) if for every open set V that contains x and every deviation d, there exists $x' \in V$ such that for every i, $u_i(d_i, x'_{-i}) \leq u_i(x')$.*

This is related to a condition in Reny [35]. A game has the single deviation property if whenever $x \in X$ is not a Nash equilibrium, there is an open set V that contains x and a deviation d such that for any $x' \in V$ there is i such that: $u_i(d_i, x'_{-i}) > u_i(x')$. Reny [35] constructed a quasiconcave compact game that satisfies the single deviation property but has no Nash equilibrium. This shows that, in contrast with games in mixed strategies, single (and so finite) deviation equilibria may fail to exist.

5.4 Tight equilibrium

To overcome the problem of non-existence above, Reny [35] introduced the following small change in the definition. He proves its existence in general topological vector spaces but, to our knowledge, he did not publish the proof yet. We provide a proof of his result in the appendix.

Definition 29 *A game G has the lower single deviation property if whenever $x \in X$ is not a Nash equilibrium, there exists $d \in X$ and a neighborhood V of x such that for every $z \in V$, there exists a player i, such that*

$$\forall x' \in V,\ \underline{u_i}(d_i, x'_{-i}) > \underline{u_i}(z)$$

This leads naturally to the following concept.

Definition 30 *A profile $x \in X$ is a lower single deviation equilibrium (LSDE) if for every $d \in X$ and every neighborhood V of x, there is $z \in V$, such that for every player i:*

$$\exists x' \in V,\ \underline{u_i}(d_i, x'_{-i}) \leq \underline{u_i}(z)$$

This notion has the advantage to not be defined on the closure of the graph. It implies Reny equilibrium: if x is a LSDE then, shrinking V to x and taking v to be a limit in the sense of nets of $u(z)$, one obtains a Reny equilibrium (x, v). The existence of a LSDE cannot be deduced from the existence of a quasi equilibrium and LSDE do not imply quasi equilibirum.

A natural question arises: is it possible to extend both ideas in one equilibrium concept? This leads us to the following new concept and result.

Theorem 31 *Every quasiconcave compact game G that has the Kakutani property admits a* ***tight equilibrium****: that is, there exists $x \in X$ such that for every neighborhood V of x and every $(\Phi_i)_{i\in N}$ in $\Pi_{i\in N} W_V(x_{-i})$, there exists $z \in V$ such that for every $i \in N$ there exists $x' \in V$ and $d'_i \in \Phi_i(x'_{-i})$ such that $u_i^{\Phi_i}(d'_i, x'_{-i}) \leq u_i(z)$, where $u_i^{\Phi}(d_i, x'_{-i})$ denotes the lower semicontinuous regularization of the restriction of u_i to the set $\{(a_i, b_{-i}) \in X_i \times V : a_i \in \Phi_i(b_{-i})\}$ at (d_i, x'_{-i}).*

Observe that this result could be applied to the game G or to its lower semicontinuous regularization $\underline{G} = ((X_i)_{i=1}^N, (\underline{u}_i)_{i=1}^N)$. Actually, if G is quasiconcave and compact, $\underline{G}$ also is (see, for example, [8] p. 11). This is of course the case almost of the results in pure strategies of our paper.

This theorem is the most technical and conceptual contribution of the paper. It implies Reny, lower-single deviation and quasi equilibrium (by shrinking V to $\{x\}$) and so strong sharing rules. Its proof is in the appendix. It is the consequence of a new selection lemma combined with Kakutani [22] fixed point theorem.

When deviation maps are taken to be constant (as in the lower single deviation equilibrium), the proof uses another fixed point theorem (Browder-Fan [32]) that does not need the TVS to be Hausdorff or locally convex: the topological assumptions needed for existence mainly depend on the structure of the deviation maps. This is not surprising.

Corollary 32 *If $G = ((X_i)_{i=1}^N, (u_i)_{i=1}^N)$ has the lower single deviation property then the set of tight equilibria of the regularized game $\underline{G} = ((X_i)_{i=1}^N, (\underline{u}_i)_{i=1}^N)$ is included in the set of Nash equilibria of G.*

Thus $\underline{G}$ admits a tight equilibrium and so G has a Nash equilibrium. Consequently:

Corollary 33 *[Reny [35]] Every quasiconcave compact game G that has the lower single deviation property admits a Nash equilibrium.*

Theorem 58 allows us to combine Barelli-Soza's result (corollary 26) and Reny's result (corollary 33) as follows.

Definition 34 *A game has the* <u>*generalized*</u> *deviation property if whenever x is not a Nash equilibrium, there is a neighborhood V of x and $(\Phi_i)_{i\in N}$ in $\Pi_{i\in N} W_V(x_{-i})$ such that for every $z \in V$ there is i, there is $x' \in V$ and $d'_i \in \Phi_i(x'_{-i})$ such that $u_i^{\Phi_i}(d'_i, x'_{-i}) > u_i(z)$.*

Theorem 35 *Every quasiconcave compact game G that has the Kakutani and the generalized deviation property admits a Nash equilibrium.*

5.5 Weak equilibrium

To overcome non-existence of single deviation equilibrium, another slight change in the definition leads us to the following concept.

Definition 36 *A strategy profile $x \in X$ is a weak equilibrium if for every open set V that contains x, every player i and every deviation $d_i \in X_i$, there exists $x' \in V$ such that $u_i(d_i, x'_{-i}) \leq u_i(x')$.*

Stated differently, x is a weak equilibrium if no player i has a robust deviation, meaning that he has no deviation d_i such that $u_i(d_i, x'_{-i}) > u_i(x')$ for all x' in some neighborhood of x. This would be a standard Nash equilibrium of the game where each player believes that the other players can do mistakes, the player has no idea about how mistakes happen, and so preserves himself against the worst possible scenario.

The difference with single deviation equilibrium is sharp: the perturbation x' is now depending on the identity of the player. Surprisingly:

Theorem 37 *Any quasiconcave compact game G admits a weak equilibrium.*

Contrary to Reny equilibrium, the proof is now a direct consequence of a standard fixed point theorem.

Proof. For every player $i = 1, ..., N$, define the multivalued mapping Φ_i from X_i to X as follows:

$$\forall x \in X,\ \Phi_i(x) = \{d_i \in X_i :\ \exists V_x \in \mathcal{V}(x) \mid \forall x' \in V_x,\ u_i(d_i, x'_{-i}) > u_i(x')\}.$$

For every player i, Φ_i has open fibers (i.e., for every $d_i \in X_i$, $\Phi_i^{-1}(d_i)$ is open), convex images, and $x_i \notin \Phi_i(x)$ for every $x \in X$. Thus, we obtain existence from the Browder-Fan theorem [32]. ■

Definition 38 *A game G is strongly secure if for every strategy profile $x \in X$ which is not a Nash equilibrium, there exists a player i, a deviation $d_i \in X_i$ and a neighborhood V of x, such that*

$$\forall x' \in V,\ u_i(d_i, x'_{-i}) > u_i(x')$$

Consequently:

Theorem 39 *Any strongly secure quasiconcave compact game G admits a Nash equilibrium.*

Applications

a) If for every i, u_i is upper semicontinuous in x and lower semicontinuous in x_{-i}, then G is strongly secure. This implies the Nash existence result of Dasgupta and Maskin [11].

b) If for every i, u_i is pseudo-continuous, meaning that that for every $x \in X$ and every $x_0 \in X$ such that $u_i(x_0) < u_i(x)$, $\limsup_{x' \to x_0} u_i(x') < u_i(x)$, and similarly for $-u_i$, then clearly G is strongly secure. This implies the Nash existence result of Morgan and Scalzo [29].

c) Since the Browder-Fan theorem [32] holds for any cardinality of N, a weak equilibrium exists in any quasiconcave compact game with infinitely many players. This may be used to prove existence of sequential equilibria in dynamic games. If $\mathcal{G}$ is an extensive form game with perfect recall with a denumerable number of players where at each information set, the player who plays at that information set has finitely many actions (less that some fixed bound K) and if the payoff function is continuous for each player for the product topology on the set of plays, then the agent normal form game in which a player at information set I is constrained to use each action with probability $\varepsilon > 0$ is continuous and quasiconcave. Thus, it admits a mixed Nash equilibrium. letting $\varepsilon \to 0$ yields to a perfect and thus a sequential and thus subgame perect Nash equilibrium of $\mathcal{G}$.

d) Reny and tight equilibria also exist independently on the cardinality of N (the proof in the appendix does use the finiteness of the N). Consequently, one may expect that generalized better-reply security and strong security may help to establish existence of Nash or approximated subgame perfect equilibria in discontinuous dynamics games.

Links with Reny and approximated equilibria

In a one player game, the set of weak equilibria is included in the set of approximated equilibria. The inclusion may be strict (while Reny and approximated equilibria coincide in that case).

Example 40 *Take* $X = [0, 2]$ *and* $u : [0, 2] \to \mathbb{R}$ *defined as follows:* $u(x) = x$ *for every* $x \in [0, 1)$, $u(x) = 0$ *for every* $x \in [1, 2)$ *and* $u(2) = 1$*. Then* $x = 1$ *is a approximated equilibrium of* G*, but is not a weak-equilibrium.*

The same happens in the following quasiconcave compact two player game. Moreover, the example is strongly secure but not better-reply secure.

Example 41 *Each player's strategy set is* $[0, 1]$*. Define* u_1*, the payoff function of player 1, as follows:* $u_1(1, y) = 1$ *for every* $y \in [0, 1]$*;* $u_1(x, 0) = 0$ *for every* $x \in [0, 1)$*; finally,* $u_1(x, y) = 1 - y$ *if* $x \neq 1$ *and if* $y \neq 0$*. Define* u_2*, the payoff function of player 2, by* $u_2(x, y) = 1-y$*. Then,* $(0, 0)$ *is an approximated equilibrium, since* $(0, \frac{1}{n})$ *is a* $\frac{1}{n}$*-equilibrium for every* n*. But* $(0, 0)$ *is not a weak-equilibrium. The only weak-equilibrium is* $(1, 0)$ *which is also the only Nash equilibrium of the game (so the game is strongly secure). It is not better-reply secure: take* $(x, u) = ((0, 0), (1, 1)) \in \overline{\Gamma}$*, where* $(1, 1) = \lim_{n\to\infty}(u_1(\frac{1}{n}, \frac{1}{n}), u_1(\frac{1}{n}, \frac{1}{n}))$*.*

Conversely, it is false that better-reply security implies strong security, as the following quasiconcave compact two player game proves.

Example 42 *Each player's strategy set is* $[0, 1]$*. Define* $u_1(x, y) = 0$ *if* $y \geq x$ *and* $u_1(x, y) = 1$ *otherwise. Let* $u_2(x, y) = 0$ *if* $x \geq y$ *and* $u_2(x, y) = 1$ *otherwise. The game is better reply-secure but not strongly secure at* $(0, 0)$*.*

Last, one can define the following class of games for which an approximated equilibrium exists, and which covers the case of California-Oregon location game:

Definition 43 G *is approximately strongly secure if for every* $x \in X$*, if* x *is not an approximated equilibrium then there exists a player* i*, a deviation* $d_i \in X_i$ *and a neighborhood* V *of* x*, such that*

$$\forall x' \in V, \ u_i(d_i, x'_{-i}) > u_i(x')$$

Theorem 44 *Any approximately strongly secure quasiconcave compact game* G *admits an approximated equilibrium.*

The advantage of weak-equilibrium and strong security are their simplicity. The drawback, compared to Reny equilibrium, is the absence of a limit payoff vector associated with the weak equilibrium strategy profile. Moreover, the following example shows that he has no predicting power when the set of discontinuities is a dense set. Of course, this never occurs in applications.

Example 45 *There is only one player. Suppose* $X = [0, 1]$ *and* $u(x) = 2x$ *for* x *rational and* $u(x) = 0$ *when* x *is irrational. The set of weak equilibria is* $[0, 1]$ *while a rational player must choose some rational number close to 1 and expect to obtain a payoff close to 2.*

5.6 Light equilibrium for symmetric games

For symmetric games, most of the ideas could be combined in one to obtain the following weak condition for existence of a symmetric Nash equilibrium: whenever a player has a profitable deviation, he has locally a regular profitable deviation map (when the other players are restricted to symmetric profiles).

Define a (quasi) symmetric game as follows: for every players i, j, assume $X_i = X_j$ (denoted Y hereafter). Moreover, for every $x, y \in Y$, assume $u_1(x, y, ..., y) = u_2(y, x, y, ..., y) = ... = u_N(y, ..., y, x)$ (denoted $u(x, y, ..., y)$ hereafter). For $y \in Y$ let $\mathbf{y} \in X$ be the symmetric profile $(y, y, ...y)$ and similarly for $\mathbf{y_{-i}} \in X_{-i}$.

A symmetric game is diagonally quasi-concave if Y is convex, and for every player i, every $x^1, ..., x^m \in Y$ and every $\bar{x} \in co\{x^1, ..., x^m\}$,

$$u(\bar{\mathbf{x}}) \geq \min_{1 \leq k \leq m} u(x^k, \bar{\mathbf{x}}_{-i}).$$

Finally, $\mathbf{y} = (y, ..., y) \in X$ is a symmetric Nash equilibrium if for every $d \in Y$, $u(d, \mathbf{y_{-i}}) \leq u(\mathbf{y})$.

Theorem 46 *Every diagonally quasi-concave symmetric compact game that has the Kakutani property, admits a **light** equilibrium $\mathbf{y} = (y, ..., y)$, that is, for every V_y a neighborhood of $y \in Y$ and every Kakutani correspondence Φ from V_y to Y, there is $z \in V_y$ and $d \in \Phi(z)$ such that $u(d, \mathbf{z_{-i}}) \leq u(\mathbf{z})$.*

The fact that there is only one utility function to consider (no quantifier with respect to the identity of player i is needed), and since the same z in V_y is present is the two sides of the equilibrium inequality, a light equilibrium refines symmetric tight equilibrium, symmetric weak equilibrium and symmetric single deviation equilibrium. Also, shrinking V_y to $\{y\}$ and considering a limit point v of $u(z)$, one obtains a symmetric quasi equilibrium which is in particular a symmetric Reny equilibrium.

Our proof is short and proceeds by contradiction, then it concatenates the local deviation maps, to obtain a global Kakutani deviation map whose fixed point provides a contradiction. A similar idea is used to prove existence of a tight.

Proof. By contradiction, suppose no light equilibrium exits. Then, for every $y \in Y$, there exists V_y a neighborhood of y and a Kakutani correspondence Φ from V_y to Y such that such for all $z \in V_y$ and all $d \in \Phi(z)$, $u(d, \mathbf{z_{-i}}) > u(\mathbf{z})$.

From the compactness of Y, there exists a finite subset $\{y_1, ..., y_K\}$ of Y, and for every $k = 1, ..., K$, there exists a Kakutani correspondence Φ_k from some neighborhood V_{y_k} of y_k to Y and where V_{y_1}, ..., V_{y_K} is a finite covering of Y. Let $\beta_1, ..., \beta_K$ be a partition of unit subordinate to the covering V_{y_1}, ..., V_{y_K}. Each β_k is a continuous function from Y in $[0, 1]$, with support in V_{y_k} and $\sum_{k=1}^K \beta_k = 1$. Define $\Phi(z) = \sum_{k=1}^K \beta_k(z)\Phi_k(z)$. It is a Kakutani correspondence from Y to it self. Thus, it has a fixed point z. Thus, there is for each k where $\beta_k(z) > 0$, $z_k \in \Phi_k(z)$ such that $z = \sum_{k=1}^K \beta_k(z)z_k$. But, $u(z_k, \mathbf{z_{-i}}) > u(\mathbf{z})$, and by diagonally quasi-concavity, one deduces that $u(\mathbf{z}) > u(\mathbf{z})$: a contradiction. ■

Consequently:

Theorem 47 *Let G be a diagonally quasi-concave symmetric compact game that has the Kakutani property. Suppose that for every symmetric profile $\mathbf{y} = (y, ..., y)$ which not a Nash (resp. approximated) equilibrium there is V_y a neighborhood of $y \in Y$, there is a Kakutani correspondence Φ from V_y to Y, such that for every $z \in V_y$ and every $d \in \Phi(z)$, $u(d, \mathbf{z_{-i}}) > u(\mathbf{z})$. Then G admits a symmetric Nash (resp. approximated) equilibrium.*

Thus, in symmetric games, Reny [34] BRS condition could be improved and proved much more directly than Reny did.

5.7 Generalized weak equilibrium

A natural question arises. Could weak equilibrium (based on single deviations) be extended to maps deviations, as it is the case in quasi, thigh and light equilibria?

Definition 48 *A strategy profile $x \in X$ is a generalized weak equilibrium if for every open set V that contains x, every player i and every Kakutani mapping Φ_i from V to X_i, there exists $x' \in V$ and $d_i \in \Phi_i(x')$ such that $u_i(d_i, x'_{-i}) \leq u_i(x')$.*

Hence, x is a generalized weak equilibrium if no player i has a regular deviation map in the neighborhood of x.

Theorem 49 *Any quasiconcave compact game G that has the Kakutani property admits a generalized weak equilibrium.*

The proof is the same as in light equilibrium but with one addition difficulty: we should deal with many utility functions, instead of one.

Proof. By contradiction, suppose no generalized weak equilibrium exits. Then, for every $x \in X$, there exists V_x a neighborhood of x, there is a player i and a Kakutani mapping Φ_i from V_x to X_i such that such for all $z \in V_x$ and all $d_i \in \Phi_i(z)$, $u_i(d_i, z_{-i}) > u(z)$.

From the compactness of X, there exists $\{x_1, ..., x_K\}$ in X, there is K players $i_1, ..., i_K$ and Kakutani mappings Φ_{i_k} from some neighborhood V_{x_k} of x_k to X_{i_k} such that V_{x_1}, ..., V_{x_K} is a finite covering of X. Let $\beta_1, ..., \beta_K$ be a partition of unit adapted to the covering V_{x_1}, ..., V_{x_K}. For each k, extend Φ_{i_k} (defined from V_{x_k} to X_{i_k}) to a correspondence Ψ_{i_k} defined from V_{x_k} to X as follows:

$$\Psi_{i_k}(x) = \Phi_{i_k}(x) \times \{x_{-i}\}.$$

Define now $\Psi(x) = \sum_{k=1}^{K} \beta_k(x)\Psi_{i_k}(x)$. It is a Kakutani correspondence from X to it self: it has a fixed point z. Thus, there is $d_{i_k} \in \Phi_{i_k}(z)$ such that $z = \sum_{k=1}^{K} \beta_k(z)(d_{i_k}, z_{-i_k})$. Consequently, for each player $i = i_k$ for some k, one has $(z_i, z_{-i}) = \sum_{k:i=i_k} [\frac{\beta_k(z)}{\sum_{k:i=i_k} \beta_k(z)}](d_{i_k}, z_{-i})$. But, $u_i(d_{i_k}, z_{-i}) > u(z)$, and by quasi-concavity, one deduces that $u(z) > u(z)$: a contradiction. ■

This leads us to the following class of games.

Definition 50 *A game G admits the regular deviation property if whenever x is not a Nash equilibrium, there is a neighborhood V of x, a player i and a Kakutani deviation mapping Φ_i from V to X_i such that, for any $y \in V$ and any $d_i \in \Phi(y)$, $u_i(d_i, y_{-i}) > u_i(y)$.*

Consequently.

Theorem 51 *Any quasiconcave compact game G that has the Kakutani and the regular deviation properties admits a Nash equilibrium.*

Observe that the same statement could be made with approximated equilibria.

5.8 Pseudo equilibrium

Single deviation equilibrium is much too strong: it may not exist. On the other hand, quasi equilibrium is a refinement that always exists. It improves Reny equilibrium by asking that, instead of a deviation d_i that works well for all x' close to x, the deviation d_i may depend on x' in a regular manner.

Recall that (x, v) is a quasi equilibrium 25, if and only if, for every i:

$$\sup_{V \in \mathcal{V}(x_{-i})} \sup_{\Phi_i \in W_V} \inf_{x'_{-i} \in V, d_i \in \Phi_i(x'_{-i})} u_i(d_i, x'_{-i}) \leq v_i,$$

where W_V is the set of multi-valued mapping from V to X_i with closed graph and non-empty convex values. Is it possible to skip this regularity requirement on Φ_i (meaning not necessarily with closed graph and or convex values)? The following analysis shows it is not. The proof in the appendix also shows why such assumption is somehow necessary to be able to apply at some step the Kakutani [22] fixed point theorem.

Recall that $(x, v) \in \overline{\Gamma}$ is a Reny equilibrium if and only if, for every player i one has:

$$\sup_{V\in\mathcal{V}(x_{-i})} \sup_{d_i\in X_i} \inf_{x'_{-i}\in V} u_i(d_i, x'_{-i}) \leq v_i,$$

where $\mathcal{V}(x_{-i})$ denotes the set of neighborhoods of x_{-i}.

One possible improvement would be by exchanging the supremum and infimum above. This leads to the following concept.

Definition 52 *$(x, v) \in \overline{\Gamma}$ is a pseudo equilibrium if for every player i:*

$$\sup_{V\in\mathcal{V}(x_{-i})} \inf_{x'_{-i}\in V} \sup_{d_i\in X_i} u_i(d_i, x'_{-i}) \leq v_i,$$

or equivalently, if $\underline{V_i}(x_{-i}) \leq v_i$, where $\underline{V_i}$ is the lower semicontinuous regularization of $V_i(x_{-i}) = \sup_{d_i\in X_i} u_i(d_i, x'_{-i})$.

Let us show that pseudo equilibrium is a quasi equilibrium without the smoothness condition.

To see why, let us prove that the quantity $\alpha_i = \inf_{x'_{-i}\in V} \sup_{d_i\in X_i} u_i(d_i, x'_{-i})$ is equal to $\beta_i = \sup_{\Phi_i\in\Omega_V} \inf_{d_i\in\Phi_i(x'_{-i}), x'_{-i}\in V} u_i(d_i, x'_{-i})$. Where Ω_V is the set of multi-valued mapping from V to X_i.

Since the maximizer could use functions instead of correspondences, one obtains that

$$\beta_i \geq \sup_{\phi_i:V\to X_i} \inf_{x'_{-i}\in V} u_i(\phi_i(x'_{-i}), x'_{-i}).$$

The right hand side quantity is easily shown to be equal to α_i. Thus, $\beta_i \geq \alpha_i$. The converse inequality is trivial because $u_i(d_i, x'_{-i}) \leq \sup_{d'_i\in X_i} u_i(d'_i, x'_{-i})$.

Consequently, the unique difference between pseudo equilibrium and quasi equilibrium is the regularity of Φ_i (closed graph and non-empty convex values). Without regularity, existence fails as the following example shows.

Example 53 *Consider a quasiconcave compact game with two players. Each player's strategy set is $[0, 1]$. Let u_1 be 1 on the diagonal and zero otherwise. Let u_2 be 1 on the anti-diagonal and zero otherwise except for the profile $(1/2, 1/2)$ where u_2 is zero and the profile $(1/2, 1)$ where it is 1. This game has no peudo equilibrium. Actually, $V_i(x_{-i}) = 1$ for both players: a continuous function. The unique possible limit vector payoffs are $v = (0, 1)$, $v = (1, 0)$ or $v = (0, 0)$. Thus, $V_i(x_{-i}) \leq v_i$ is impossible for both players.*

As is shown in the appendix, we are constrained to use deviation maps $\{\Phi_i\}$ that allow the use of a fixed point theorem. We explore two of them: maps that have a closed graph with non-empty convex values (in that case one needs the Kakutani property to hold) and constant maps (and then no further assumption is needed on the topological vector space). A third possibility of deviation maps under which a relaxed equilibrium will exist is by using continuous functions instead of correspondences. They are also more easy to check in practice. In that case, one needs the convex compact set X to satisfy the fixed point property (meaning that any continuous function on X needs to admit a fixed point). Recently, Cauty [10] proves the Schauder's conjecture that states that such a property holds when the TVS is Hausdorff (not necessarily locally convex).

6 Conclusion

For games in mixed strategies, finite deviation equilibrium is easily shown to exist and it refines Reny, single deviation, weak and Simon and Zame equilibrium. This marriage is impossible in pure strategies (many of those notions do not exist or differ). Three questions arise: Does pseudo equilibrium in mixed strategies always exist? Could our solution concepts extend to *non* quasiconcave compact games in pure strategies? Is it possible to refine finite deviation equilibrium in mixed strategies for example to cover tight equilibrium or generalized weak equilibrium?

To the first question, the answer is negative. Actually, if G is a zero-sum game, pseudo equilibria exist if and only if the game has a value. Since there are games in pure or mixed strategies without a value, a pseudo equilibrium may not exist. The proof follows. Let S (resp. T) denote the compact set of strategies of player 1 (resp. 2) and let $f = u_1 = -u_2$ be a bounded payoff function. Let $((s,t),v) \in \overline{\Gamma}$ be a pseudo equilibrium. Then, there exists s_n that converges to s and t_n that converges to t such that:

$$\lim_n \sup_{s' \in S} f(s', t_n) \leq v \leq \lim_n \inf_{t' \in T} f(s_n, t'),$$

consequently:

$$\inf_{t' \in T} \sup_{s' \in S} f(s', t') \leq v \leq \sup_{s' \in S} \inf_{t' \in T} f(s', t').$$

That is, the game has a value. The converse is trivial.

To the second question (definition and existence of a tight equilibrium for non quasiconcave compact games in pure strategies and their sharing rule interpretation), the answer is positive. Moreover, this extends Bich [5] and Mclennan et al [27]. For the clarity of the presentation, this is solved in a subsequent paper.

For the last question, observe that any compact-metric game in mixed strategies admits two different refinements of Reny equilibrium: tight equilibrium and finite deviation equilibrium. The first is constructed with respect to robustness against all deviations maps that react in a regular manner to small perturbations of the opponents strategies. The second is constructed with respect to robustness against all deviations in finite discretizations of the game, but without the possibility to react to perturbations of the opponents strategies. The two ideas are different. We are trying to reconcile both of them in one concept.

However, the FDE could be improved strictly by allowing the finite set of deviations D_i's in its definition to contain mixed strategies (formally by taking D_i as a finite subset of M_i rather of X_i). The proof of existence is exactly the same.

The challenge now is to confront the new conditions to the variety of potential applications.

Appendix

A) More on approximated equilibria

Definition 54 *A game $G = ((X_i)_{i \in N}, (u_i)_{i \in N})$ is weakly payoff secure at $(x,v) \in \overline{\Gamma}$ if for every $\epsilon > 0$, for every neighborhood $W_x \times W_v$ of (x,v), there exists $(x', u(x')) \in W_x \times W_v$ such that for every $d \in X$, every player i can secure a payoff strictly above $u_i(d_i, x'_{-i}) - \epsilon$ at x.*

Let us now formulate approximately better-reply security condition in the spirit of Reny's theorem.

A game is called *weakly better-reply secure* if for every $(x,v) \in \overline{\Gamma}$ such that x is not an approximated equilibrium, either: [a] some player i can secure a payoff strictly above v_i, or [b] the game is weakly payoff secure at (x,v).

Lemma 55 *Weakly better-reply security is equivalent to approximately better-reply security.*

Proof. Let G weakly better-reply secure. We prove that G is approximately better-reply secure, or equivalently, that if $(x,v) \in \overline{\Gamma}$ is a Reny equilibrium, then x is an approximated equilibrium. By contradiction, assume that $(x,v) \in \overline{\Gamma}$ is a Reny, so satisfies $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i$ for every player i, and that it is not an approximated equilibrium. Thus, condition [a] is false, so that [b] must be true: the game is weakly payoff secure at (x,v). Now, let $\epsilon > 0$. Define $W_x \times W_v$ to be an open neighborhood of (x,v), where $W_v = \{v' \in \mathbb{R}^N : \forall i \in N, |\ \bar{v}_i - v'_i\ | < \frac{\epsilon}{2}.\}$. Since G is weakly payoff secure at (x,v), there exists $(x', u(x')) \in W_x \times W_v$, there exists $d' \in X$ such that for every player i,

$\underline{u_i}(d'_i, x_{-i}) \geq \sup_{d_i} u_i(d_i, x'_{-i}) - \frac{\epsilon}{2}$. Now, since (x, v) is a Reny equilibrium, $\sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \leq v_i$ for every $i \in N$. Finally, $v_i \geq \sup_{d_i \in X_i} \underline{u_i}(d_i, x_{-i}) \geq \sup_{d_i \in X_i} u_i(d_i, x'_{-i}) - \frac{\epsilon}{2}$; consequently, $u_i(x') \geq v_i - \frac{\epsilon}{2} \geq \sup_{d_i \in X_i} u_i(d_i, x'_{-i}) - \epsilon$, which proves that x is an approximated equilibrium, a contradiction.

Conversely, if G is approximately better-reply secure, then let $(x, v) \in \overline{\Gamma}$ such that x is not an approximated equilibrium. Thus, it is not a Reny equilibrium (from approximately better-reply secure assumption), which implies condition [a] of weak better-reply secure assumption. ■

Let us apply this characterization to the following zero-sum timing game.

Example 56 *Each player's strategy set is $[0, 1]$. Let $u_1(x, y) = e^{-\min\{x,y\}}$ for every $y \neq x$ and $u_1(x, x) = 0$. Let G be the mixed extension of this game. It is a quasi-concave compact game. To see that this game is not better-reply secure, take the sequence of mixed strategies (σ_1^n, σ_2^n), where σ_1^n is uniform on $[0, \frac{1}{n}]$ and $\sigma_2^n = 0$. Then $(\sigma_1^n, \sigma_2^n, u_1(\sigma_1^n, \sigma_2^n), u_2(\sigma_1^n, \sigma_2^n))$ tends to $(0, 0, 1, -1)$ which is a Reny equilibrium. However, $(0, 0)$ is not a Nash equilibrium. Now, prove that G is weakly better-reply secure. Assume $(\sigma_1, \sigma_2, v_1, v_2) \in \overline{\Gamma}$ and (σ_1, σ_2) is not a Nash (and so an approximate) equilibrium.*

- *If $v_1 < 1$, then player 1 can secure a payoff $1 > v_1$ at σ, playing a uniform random variable on $[0, 1]$. Condition [a] of weak better-reply security assumption holds.*
- *If $v_1 = 1$, then $v_2 = -1$ and $\sigma_1 = 0$. To prove that G is weakly payoff secure at σ, prove that condition [b] holds, i.e. that the game is weakly payoff secure at $(\sigma_1, \sigma_2, v_1, v_2)$. Take $\epsilon > 0$, and a neighborhood $W_\sigma \times W_v$ of (σ, v). Take σ_1^n uniform on $[0, \frac{1}{n}]$ and $\sigma_2^n = \sigma_2$. For n large enough, $(\sigma^n, u(\sigma^n)) \in W_\sigma \times W_v$, $\underline{u_1}(\sigma_1^n, \sigma_2) > 1 - \epsilon$ and $\underline{u_2}(\sigma_1, \sigma_2^n) > -1 - \epsilon$. Consequently, each player i can secure a payoff strictly above $\sup_{d_i \in X_i} u_i(d_i, \sigma_{-i}^n) - \epsilon$ at σ (by playing σ_i^n).*

To conclude, observe that this game is not payoff secure at $(0, 0)$ since player 2 cannot secure a payoff above $-\epsilon$ for every $\epsilon > 0$. Consequently, corollary 44 can be applied, but not Prokopovych [33].

B) Selection lemma

In the following lemma, for every $y = (y_i)_{i \in N} \in \mathbb{R}^N$ and every $y' = (y'_i)_{i \in N} \in \mathbb{R}^N$, denote $y \vee y' = (\max\{y_i, y'_i\})_{i \in N} \in \mathbb{R}^N$. A multivalued mapping Ψ from X to $\mathbb{R}^N$ is $\vee$-stable if for every $x \in X$ and every $(y, y') \in \Psi(x) \times \Psi(x)$, one has $y \vee y' \in \Psi(x)$.

Lemma 57 *Let X be a compact subset of a topological vector space. Let Ψ be a $\vee$-stable multivalued mapping from X to $\mathbb{R}^N$ such that for every $x \in \mathbb{R}^N$, there exists a neighborhood V of x with $\cap_{x' \in V} \Psi(x') \neq \emptyset$. Then Ψ admits a selection $\psi = (\psi_i)_{i \in N}$ (where ψ_i is the i-th component of ψ) such that for every $i \in N$ and every $\alpha \in \mathbb{R}^N$, the set $\{x \in X : \forall i \in N,\ \psi_i(x) \leq \alpha_i\}$ is open in X.*

Proof. For every $x \in X$, let $V(x)$ be a compact[3] neighborhood of x such that $\cap_{x' \in V(x)} \Psi(x') \neq \emptyset$, and choose $y(x) \in \cap_{x' \in V(x)} \Psi(x')$. Since X is compact, there exist some compact neighborhoods $V(x_1), ..., V(x_n)$ of x_1, ..., x_n, whose interiors cover X. For every $x \in X$, define

$$\psi(x) = \vee_{k : x \in V(x_k)} y(x_k). \tag{2}$$

Since Ψ is $\vee$-stable, the mapping $\psi : X \to \mathbb{R}^N$ is a selection of Ψ. Now, prove that for every $i \in N$ and every $\alpha \in \mathbb{R}^N$, the set

$$Y := \{x \in X : \forall i \in N,\ \psi_i(x) \leq \alpha_i\} \tag{3}$$

is an open subset of X. Fix $\alpha \in \mathbb{R}^N$ and let $\bar{x} \in Y$. We now prove that

$$V := (\cup_{k : \bar{x} \in V(x_k)} V(x_k)) \cap (\cap_{k' : \bar{x} \notin V(x_{k'})} {}^c V(x_{k'})) \tag{4}$$

[3] Without any loss of generality, since X admits a compact basis of neighborhoods at every $x \in X$, taking a smaller neighborhood if necessary, we can assume V compact.

is a neighborhood of $\bar{x}$, and is included in Y.

First, V is clearly a neighborhood of $\bar{x}$ in X: indeed, $\bar{x}$ belongs the interior of some $V(x_k)$ (since the interiors of $V(x_1), ..., V(x_n)$ cover X), thus $\cup_{k:\bar{x}\in V(x_k)} V(x_k)$ is a neighborhood of $\bar{x}$. Moreover, $\cap_{k':\bar{x}\notin V(x_{k'})} {}^c V(x_{k'})$ is a finite intersection of (open) neighborhoods of $\bar{x}$.

Second, prove that $V \subset Y$. Fix $y \in V$. From Equation 3, proving $y \in Y$ amounts to proving that for every $i \in I$, $\psi_i(y) \leq \alpha_i$. This can be written equivalently, from the definition of ψ_i (Equation 2): for every $i \in I$, for every k such that $y \in V(x_k)$, one has $y_i(x_k) \leq \alpha_i$. To prove this last inequality, fix $i \in I$ and k such that $y \in V(x_k)$. This implies, from $y \in V$, and using Equation 4, that $\bar{x} \in V(x_k)$. Using Equation 2 at $\bar{x}$ and the fact that $\bar{x} \in V(x_k)$, one obtains $y_i(x_k) \leq \psi_i(\bar{x})$. But since $\bar{x} \in Y$, one has $\psi_i(\bar{x}) \leq \alpha_i$ which finally implies $y_i(x_k) \leq \alpha_i$.

■

C) Existence of tight equilibria

Recall that for every $x_{-i} \in X_{-i}$ and every neighborhood V of x_{-i}, $W_V(x_{-i})$ denotes the set of multi-valued mapping from V to X_i with closed graph and non-empty convex values. Then, for every player i, every $x'_{-i} \in V$ and $d'_i \in \Phi_i(x'_{-i})$, $u_i^{\Phi}(d'_i, x'_{-i})$ denotes the lower semicontinuous regularization of the restriction of u_i to the set $\{(a_i, b_{-i}) \in X_i \times V : a_i \in \Phi_i(b_{-i})\}$ at (d'_i, x'_{-i}). Denote $\mathcal{N}(x)$ the set of neighborhood of x. If V is a neighborhood of x, denote V_{-i} the neighborhood of x_{-i} obtained by projecting V on X_{-i}.

We want to prove:

Theorem 58 *Every quasiconcave compact game G admits a* ***tight equilibrium****: that is, there exists $x \in X$ such that for every neighborhood V of x and every $(\Phi_i)_{i\in N}$ in $\Pi_{i\in N} W_V(x_{-i})$, there exists $z \in V$ such that for every $i \in N$ there exists $x' \in V$ and $d'_i \in \Phi_i(x'_{-i})$ such that $u_i^{\Phi_i}(d'_i, x'_{-i}) \leq u_i(z)$.*

We prove Theorem 58 by contradiction: assume G has no tight equilibrium. Define the multivalued mapping Ψ from X to $\mathbb{R}^N$ as follows: for every strategy profile $x \in X$,

$$\Psi(x) = \{\alpha \in \mathbb{R}^N : \exists V \in \mathcal{N}(x),\ \exists (\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i}) : (1) \inf_{x'_{-i}\in V_{-i},\ d'_i\in\Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \alpha_i$$

$$(2)\ \forall z \in V,\ \exists i_0 : \alpha_{i_0} > u_{i_0}(z)\}.$$

We now check that Lemma 57 can be applied to Ψ.

Step 1: Ψ has non-empty values: indeed, by assumption, for every $x \in X$, x is not a tight equilibrium: by contraposition of the definition of a tight equilibrium, there exists a neighborhood (which can be assumed to be compact) V of x and $(\Phi_i)_{i\in N}$ in $\Pi_{i\in N} W_V(x_{-i})$ such that for every $z \in V$ there exists $i_0 \in N$ such that for every $x' \in V$ and $d'_{i_0} \in \Phi_{i_0}(x'_{-i_0})$, one has

$$u_{i_0}^{\Phi_{i_0}}(d'_{i_0}, x'_{-i_0}) > u_{i_0}(z). \tag{5}$$

For every $i \in N$, define $\alpha_i = \inf_{x'_{-i}\in V_{-i},\ d'_i\in\Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i})$, so that Condition (1) above is satisfied. Moreover, since the restriction of u_i^{Φ} to the compact set $\{(d'_i, x'_{-i}) \in X_i \times V_{-i} : d'_i \in \Phi_i(x'_{-i})\}$ is lower semicontinuous, $\alpha_i = u_i^{\Phi_i}(d'_i, x'_{-i})$ for some $x'_{-i} \in V_{-i}$ and $d'_i \in \Phi_i(x'_{-i})$. Thus, for every $z \in V$, from Inequality 5, one has $\alpha_{i_0} > u_{i_0}(z)$ for some $i_0 \in N$, i.e. Condition (2) in the definition of $\Psi(x)$ is also satisfied.

Step 2: Ψ is a $\vee$-stable multivalued mapping: take $x \in X$, α and α' in $\Psi(x)$. From the definition of $\Psi(x)$, there exists $V \in \mathcal{N}(x)$ (resp. there exists $V' \in \mathcal{N}(x)$), and there exists $(\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i})$ (resp. $(\Phi'_i)_{i\in N} \in \Pi_{i\in N} W_{V'_{-i}}(x_{-i})$), both satisfying:

$$\inf_{x'_{-i}\in V_{-i},\ d'_i\in\Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \alpha_i \tag{6}$$

$$\forall z \in V,\ \exists i_0 : \alpha_{i_0} > u_{i_0}(z) \tag{7}$$

$$\inf_{x'_{-i} \in V'_{-i},\ d'_i \in \Phi'_i(x'_{-i})} u_i^{\Phi'_i}(d'_i, x'_{-i}) \geq \alpha'_i \tag{8}$$

$$\forall z \in V',\ \exists i_0 : \alpha'_{i_0} > u_{i_0}(z) \tag{9}$$

For every $i \in N$, define $\Phi''_i \in W_{V_{-i}}(x_{-i})$ as follows: $\Phi''_i(x) = \Phi_i(x)$ if $\alpha_i \geq \alpha'_i$ and $\Phi''_i(x) = \Phi'_i(x)$ otherwise. Define $V'' = V' \cap V$ and $\alpha'' = \alpha \vee \alpha'$. To prove that $\alpha'' \in \Psi(x)$, prove that Φ'' and V'' is suitable in the definition of $\alpha'' \in \Psi(x)$. First, from Inequation 7 or Inequation 9, one has: for every $z \in V''$, there exists i_0 such that $\alpha''_{i_0} > u_{i_0}(z)$, which proves the second condition of $\alpha'' \in \Psi(x)$. Second, for every $i \in N$, if $\alpha_i \geq \alpha'_i$, one has

$$\inf_{x'_{-i} \in V''_{-i},\ d'_i \in \Phi''_i(x'_{-i})} u_i^{\Phi''_i}(d'_i, x'_{-i}) = \inf_{x'_{-i} \in V''_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \inf_{x'_{-i} \in V_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \alpha_i = \alpha''_i$$

and the case $\alpha_i < \alpha'_i$ is similar. Thus, one has obtained $\alpha'' = \alpha \vee \alpha' \in \Psi(x)$, i.e. Ψ is a $\vee$-stable multivalued mapping.

Step 3 : Ψ has open pre-images: this is simply because the two conditions on α in the definition of $\Psi(x)$ have to be true only one some neighborhood of x.

Thus, we can apply Lemma 57. This provides a selection $\alpha : X \to \mathbb{R}^N$ of Ψ, which satisfies the openness condition of Lemma 57. To finish the proof of Theorem 58, we apply a fixed-point theorem à la Kakutani to the following multivalued mapping Ψ'. Define, for every $x \in X$:

$$\Psi'(x) = co\{d \in X : \exists V \in \mathcal{N}(x) \text{ and } (\Phi_i)_{i \in N} \in \Pi_{i \in N} W_{V_{-i}}(x_{-i})$$
$$\text{such that for every } i \in N,\ (d_i)_{i \in N} \in \Pi_{i \in N}\Phi_i(x_{-i}) \text{ and } \inf_{x'_{-i} \in V_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \sup_{x'' \in V} \alpha_i(x'')\}.$$

First, Ψ' admits a local multivalued selection $\Pi_{i \in N}\Phi_i(x_{-i})$ at every x, this selection having a closed graph, and nonempty and convex values. Indeed, for every $x \in X$, from the definition of α, there exists $V \in \mathcal{N}(x)$ and $\Phi = (\Phi_i)_{i \in N} \in \Pi_{i \in N} W_{V_{-i}}(x_{-i})$ such that for every $i \in N$ one has the following inequality: $\inf_{x'_{-i} \in V_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \alpha_i(x)$. But from the property of α given by Lemma 57, the set $\{x \in X : \inf_{x'_{-i} \in V_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i})) \geq \alpha_i(x)\}$ is open, consequently $\Pi_{i \in N}\Phi_i(x_{-i}) \subset \Psi'(x)$ (taking V smaller if necessary).

Second, Ψ' has convex images, since it is defined as a convex hull.

Consequently, we can apply the following generalization of Kakutani's theorem to Ψ': in the following, we say that a multivalued mapping F from X to X (where X is a topological vector space) is admissible if it has nonempty, compact, convex values and has a closed graph. We say that a topological vector space X has the Kakutani's property if every admissible mapping F from X to X has a fixed point.

Theorem 59 *Let X be a compact subspace of a topological space, with the Kakutani's property. Let F from X to X with convex values, such that for every $x \in X$, there exists a multivalued admissible mapping G on a neighborhood $V_x \subset X$ of x into X, such that for every $x' \in V_x$, $G(x) \subset F(x)$ (G is called a local multivalued selection of F). Then F admits a fixed point.*

Proof. From the compactness of X, there exists a finite subset $\{x_1, ..., x_K\}$ of X, and for every $k = 1, ..., K$, there exists G_k an admissible multivalued selection of F from some neighborhood V_{x_k} of x_k to X, where V_{x_1}, ..., V_{x_K} is a finite covering of X. Let $\beta_1, ..., \beta_K$ a partition of unit subordinate to the covering V_{x_1}, ..., V_{x_K}. Thus, each β_k is a continuous function from X in $[0, 1]$, with support in V_{x_k}, such that for every $x \in X$, $\sum_{k=1}^K \beta_k(x) = 1$. Define $G(x) = \sum_{k=1}^K \beta_k(x)G_k(x)$. It is clearly an admissible multivalued selection of F (because F has convex values), so admits a fixed point, which is a fixed point of F.

■

Applying the fixed-point theorem above, Ψ' admits a fixed point $\bar{x} \in X$. This means that there exists $d(1), ..., d(K)$ in X, $\lambda(1) \geq 0$, ...,$\lambda(K) \geq 0$ with $\sum_{k=1}^{K} \lambda(k) = 1$, such that $\bar{x} = \sum_{k=1}^{K} \lambda(k)d(k)$ and such that for every $k = 1, ..., K$, there is some $V \in \mathcal{N}(x)$ and some $(\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i})$ such that $u_i(d_i(k), \bar{x}_{-i}) \geq u_i^{\Phi_i}(d_i(k), \bar{x}_{-i}) \geq \alpha_i(\bar{x})$, the first inequality being a consequence of the definition of $u_i^{\Phi_i}$.

But the mapping $u_i(., \bar{x}_{-i})$ is quasi concave. Consequently, from the above inequalities for every $k = 1, ..., K$, one obtains $u_i(\bar{x}) \geq \alpha_i(\bar{x})$ for every $i \in N$, which yields a contradiction with Condition (2) in the definition of Ψ, i.e. the fact that for every z in some neighborhood V' of $\bar{x}$, there exists $i_0 \in N$ such that $\alpha_{i_0}(\bar{x}) > u_{i_0}(z)$.

D) Existence of the lower single deviation equilibrium

A profile $x \in X$ is a weak single deviation equilibrium if for every $d \in X$ and every neighborhood V of x, there is $z \in V$, such that for every player i:

$$\exists x' \in V, \ \underline{u_i}(d_i, x'_{-i}) \leq u_i(z)$$

Theorem 60 *Every quasiconcave compact game G has a weak single deviation equilibrium.*

Observe that the theorem does not assume any topological assumption on the topological vector space.

This theorem implies existence of a lower single deviation equilibrium (when the theorem is applied to quasiconcave utility functions $\underline{u_i}$) and so also implies corollary 33 established by Reny [35].

The changes with respect the above proof follows.

- $(\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i})$ is replaced by $(d_i)_{i\in N} \in \Pi_{i\in N} X_i$,
  $u_i^{\Phi_i}(d_i, x_{-i})$ by $\underline{u_i}(d_i, x_{-i})$,
- $\Psi(x) = \{\alpha \in \mathbb{R}^N : \exists V \in \mathcal{N}(x), \exists (\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i}) : (1) \inf_{x'_{-i}\in V_{-i},\ d'_i \in \Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \alpha_i \ (2) \ \forall z \in V, \ \exists i_0 : \alpha_{i_0} > u_{i_0}(z)\}$, is replaced by
  $\Psi(x) = \{\alpha \in \mathbb{R}^N : \exists V \in \mathcal{N}(x), \exists (d_i)_{i\in N} \in \Pi_{i\in N} X_i : (1) \inf_{x'_{-i}\in V_{-i}} \underline{u_i}(d_i, x'_{-i}) \geq \alpha_i \ (2) \ \forall z \in V, \ \exists i_0 : \alpha_{i_0} > u_{i_0}(z).\}$
- and

$$\Psi'(x) = co\{d \in X : \exists V \in \mathcal{N}(x) \text{ and } (\Phi_i)_{i\in N} \in \Pi_{i\in N} W_{V_{-i}}(x_{-i})$$
$$\text{such that for every } i \in N, d_i \in \Phi_i(x) \text{ and } \inf_{x'_{-i}\in V_{-i}, d'_i\in\Phi_i(x'_{-i})} u_i^{\Phi_i}(d'_i, x'_{-i}) \geq \sup_{x''\in V} \alpha_i(x'')\},$$

is replaced by

$$\Psi'(x) = co\{d \in X : \exists V \in \mathcal{N}(x) \text{ and } (d_i)_{i\in N} \in \Pi_{i\in N} X_i$$
$$\text{such that for every } i \in N \text{and } \inf_{x'_{-i}\in V_{-i}} \underline{u_i}(d_i, x'_{-i}) \geq \sup_{x''\in V} \alpha_i(x'')\}.$$

Then, Ψ' has nonempty convex values, and has open pre images: Browder-Fan theorem [32] can be applied (instead of Kakutani's theorem) to obtain a fixed point of Ψ'. Actually, Browder-Fan theorem only requires X to be a compact and convex subset of a (possibly non Hausdorff) topological vector space. Consequently, one obtains the existence of a Reny equilibrium with this weaker assumption on the strategy space.

E) Infinite number of players

In sections B), C) and D) above, the finiteness of the number of players is not used: Browder-Fan and Kakutani theorems are still valid. Thus, in pure strategies, existence of tight, quasi, lower single deviation, Reny and sharing rule equilibrium hold whatever is the set of players, be it finite or not.

References

[1] Aliprantis C. D. and Border K.C. (2007). Infinite Dimensional Analysis. *Springer.*

[2] Bagh A. and Jofre A. (2006). Reciprocal upper semicontinuity and better reply secure games: a comment. *Econometrica*, **74(6)**, 1715-1721.

[3] Barelli P., Soza I. (2009). On the existence of Nash equilibria in discontinuous and qualitative games. *University of Rochester.*

[4] Baye M., Tian G. and Zhou J. (1993). Characterizations of the existence of equilibria in games with discontinuous and non-quasiconcave Payoffs. *Review of Economic studies*, **60(4)**, 935 - 948.

[5] Bich P. (2010). Existence of pure Nash equilibria in discontinuous and non quasiconcave games. *International Journal of Game Theory*, **38(3)**, 395-410.

[6] Bonnisseau J.M., P. Gourdel and H. Hammami (2009). Existence of the Nash equilibrium in discontinuous games. *C. R. Acad. Sci. Paris, Ser. I*, 347, 701-704.

[7] Carmona G. (2009). On existence result for discontinuous games. *Journal of Economic Theory*, **144(3)**, 1333-1340

[8] Carmona G. (2011). Understanding some recent existence results for discontinuous games. *Economic Theory*, **4**8 (1), 31-45.

[9] De Castro L. (2010). Equilibrium existence and approximation of regular discontinuous games. *Economic Theory*, 1-19.

[10] Cauty, R. (2001). Solution du Problème de Point Fixe de Schauder. *Fund. Math.*, **170**, 231-246.

[11] Dasgupta P. and Maskin E. (1986). The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. *The Review of Economic Studies*, **53(1)**, 1-26.

[12] Dasgupta P. and Maskin E. (1986). The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. *The Review of Economic Studies*, **53(1)**, 27-41.

[13] Fang H and Morris S. (2006). Multidimensional private value auctions. *Journal of Economic Theory*, **126**, 130

[14] Glicksberg I .(1952). A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points. *Proceedings of the American Mathematical Society*, **3**, 170-174.

[15] Hart O. (1975). On the optimality of equilibrium when the market structure is incomplete. *Journal of Economic Theory*, **11**, 418-443.

[16] M. Hirsch, Magill M. and Mas-Colell M. (1987). A geometric approach to a class of equilibrium existence theorems. *Journal of Mathematical Economics*, **19**, 95-106.

[17] Hoernig S. H (2007). Bertrand games and sharing rules. *Economic Theory*, **31(3)**, 573-585.

[18] Husseini S.Y., Lasry J.M. and Magill M. (1986). Existence of equilibrium with incomplete markets. *Journal of Mathematical Economics*, **19**, 39-67.

[19] Jackson M., Simon L.K., Swinkels J.M. and Zame W.R. (2002). Communication and equilibrium in discontinuous games of incomplete information. *Econometrica*, **70(5)**, 1711-1740.

[20] Jackson M. (2009). Non-existence of equilibrium in Vickrey, second-price, and English auctions. *Rev. Econ. Design*, **13**, 137-145

[21] Jackson M.O. and Swinkels J.M. (2005). Existence of equilibrium in single and double private value auctions *Econometrica*, **73**, No., 93-139.

[22] Kakutani S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, **8**, 416-427.

[23] Kim I.S. (2004). Fixed points of condensing multivalued maps in topological vector spaces. *Fixed Point Theory and Applications*, **2**, 107-112.

[24] Kim J. and Che Y.-K. (2004). Asymmetric information about rivals types in standard auctions. *Games Econ. Behav.* , **46**, 383-397.

[25] Laraki R., Solan E. and Vieille N. (2005). Continuous-time games of timing. *Journal of Economic Theory*, **120**, 206-238.

[26] Maskin E. and J. Riley (2000). Equilibrium in sealed high bid auctions. *Rev. Econ. Stud.*, **67**, 439-454.

[27] McLennan A., P.K. Monteiro and R. Tourky (2011). Games with discontinuous payoffs: a strengthening of Reny's existence theorem, *Econometrica*, **79(5)**, 1643-1664.

[28] Mertens J.-F., S. Sorin and S. Zamir (1994). *Repeated Games*. CORE discussion paper 9420-9422.

[29] Morgan, J. and V. Scalzo (2007). Pseudocontinuous functions and existence of Nash equilibria. *Journal of Mathematical Economics*, **43(2)**, 174-183,

[30] Nash J. (1950). Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, **36**, 48-49.

[31] Nash J. (1951). Non-cooperative games. *Annals of Mathematics*, **54**, 286-295.

[32] Park S. (2004). New versions of the Fan-Browder fixed point theorem and existence of economic equilibria. *Fixed Point Theory and Applications*, **2004(2)**, 149-158.

[33] Prokopovych P. (2011). On equilibrium existence in payoff secure games. *Economic Theory*, **48(1)**, 5-16.

[34] Reny, P.J. (1999). On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica*, **67(5)**, 1029-1056.

[35] Reny P.J. (2009). Further results on the existence of Nash equilibria in discontinuous games. *mimeo, University of Chicago.*

[36] Reny, P.J. (2011). Strategic approximations of discontinuous games. *Economic Theory*, **48**, 17-29.

[37] Simon, C. and M. Stinchcombe (1995). Equilibrium refinement for infinite normal-form games. *Econometrica*, **63**, 1421-1443.

[38] Simon, L.K. and Zame, W.R. (1990). Discontinuous games and endogenous sharing rules. *Econometrica*, **58**, 861-872.

[39] Sion M. and Wolfe P. (1957). On a game without a value. In Contibutions to the Theory of Games, III, Dresher M., A.W. Tucker and P. Wolfe (eds.). *Annals of Mathematical Studies*, **39**, Princeton University Press, 299-306.

[40] van Damme E. (1984). A relation between perfect equilibria in extensive form games and proper equilibria in normal form games. *International Journal of Game Theory*, **13**, 1-13.

[41] Vieille N. (2000a). Two-player stochastic games I: a reduction. *Israel Journal of Mathematics,* **119**, 55-91.

[42] Vieille N. (2000b). Two-player stochastic games II: the case of recursive games. *Israel Journal of Mathematics,* **119**, 93-126.